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Non-relativistic fermions and effects related to fractional charge

D Waxman

School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, UK

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Abstract. Non-relativistic fermions whose motion is constrained by an infinite potential step (hard wall) are considered. Relative to the bulk, a net fermionic charge is induced at the wall due to the vanishing of the many-particle wavefunctions. The charge, linear charge density and the fluctuations from their mean values are calculated.

1. Introduction

In the recent past it has been shown that objects having fractional fermionic charge can arise in systems containing identical fermions (each of which carries unit fermionic charge) (Jackiw and Rebbi 1976). The charge (and localisation) of these objects is sufficiently well defined (Kivelson and Schrieffer 1982, Rajaraman and Bell 1982) that as far as macroscopic considerations (i.e. measurements) go we can ignore fluctuations in the charge. This is really the key to the interest in these objects. It is trivial to find systems that on average have fractional charge. The objects studied by the above authors have fractional charge with vanishingly small mean square deviations.

Central to the attainment of the fractional charge has been the presence of topologically non-trivial solitons. It was the objective of the investigation presented here to see if entities carrying fractional charge (again with vanishingly small mean square charge deviations) could be found in much more mundane situations than fermions moving in the field of a soliton.

The system we have focused our attention on that could, quite possibly, exhibit such behaviour is rather simple. It is a system of non-relativistic fermions whose motion is restricted by a hard wall (infinite potential step). The principal idea behind this choice lies in the fact that in such systems there is generally a density 'hole' in the vicinity of the wall—resulting from the vanishing of all the one-particle wavefunctions at the wall. (A clear illustration of this point for a one-dimensional system can be seen on p 71 in the book by Peierls (1979).)

The calculation we present illustrates that while it is not possible† to find a localised fractional charge which is fluctuation free (in the sense described above) it is possible to find a linear charge density which does have this property.

† In the very simple systems considered.

2. Definition of the problem

We consider a system of identical fermions moving in two dimensions. These are spinless, non-interacting and move non-relativistically. There are N fermions per unit area.

The region of space we consider is

$$L_x \geqslant x \geqslant 0 \qquad L_y \geqslant y \geqslant 0. \tag{2.1}$$

We impose hard wall boundary conditions at $x=0$ and $x=L_x$ and periodic boundary conditions along the y direction. Thus if the many-particle wavefunction is constructed out of the one-particle wavefunctions $\phi(x, y)$ these satisfy

$$\phi(0, y)=\phi(L_x, y)=0 \tag{2.2}$$

$$\phi(x, y)=\phi(x, y+L_y). \tag{2.3}$$

The quantities which we shall calculate with respect to the ground state of the system are

(1) the additional (average) fermionic charge/unit length associated with the wall at $x=0$;

(2) the mean square deviation of the charge/unit length from its average value.

In order to define the additional fermionic charge/unit length it is necessary to make a comparison with a reference system. The reference system we adopt is the one with the same number N of fermions/unit area but with periodic boundary conditions along *both* x and y directions.

3. Preliminaries†

For the system with walls let a_{np} denote the fermion annihilation operator for the one-particle states. These are

$$\langle x, y|\, a^+_{np}\,|\text{vac}\rangle = \phi_{np}(x, y) \tag{3.1}$$

$$= (1/L_y)^{1/2}\, \mathrm{e}^{\mathrm{i}py}\, \phi_n(x) \tag{3.2}$$

with

$$\phi_n(x) = \left(\frac{2}{L_x}\right)^{1/2} \sin\left(\frac{n\pi x}{L_x}\right) \qquad n=1, 2, 3, \ldots. \tag{3.3}$$

Conditions (2.3) imply

$$pL_y = 2\pi n \qquad n = 0, \pm 1, \pm 2 \ldots. \tag{3.4}$$

The ground state of this system consists of all one-particle states up to the Fermi energy ε_{F} being filled:

$$|gd\rangle = \left(\prod_{\varepsilon_{np}<\varepsilon_{\mathrm{F}}} a^+_{np}\right)|\text{vac}\rangle \tag{3.5}$$

with

$$\varepsilon_{np} = \frac{1}{2m}\left[\left(\frac{n\pi}{L_x}\right)^2 + p^2\right]. \tag{3.6}$$

† We set $\hbar = 1$ throughout.

In order to consider the additional charge/unit length it is necessary to define the appropriate operator that represents this quantity.

Defining the field operator $\psi(\boldsymbol{r})$ in terms of the eigenfunctions of (3.1) we have

$$\psi(\boldsymbol{r})=\sum_{np}\phi_{np}(\boldsymbol{r})a_{np} \qquad \boldsymbol{r}\doteq(x,y). \tag{3.7}$$

We specify the Fermi energy by requiring

$$NL_xL_y=\int_0^{L_x}\mathrm{d}x\int_0^{L_y}\mathrm{d}y\langle\psi^+(\boldsymbol{r})\psi(\boldsymbol{r})\rangle \tag{3.8}$$

the expectation value taken with respect to the ground state. Thus the operator representing the density deviation is

$$\rho(\boldsymbol{r})=\psi^+(\boldsymbol{r})\psi(\boldsymbol{r})-N \tag{3.9}$$

with the subtraction N corresponding to the average fermion density in the wall-free system†.

The operator representing the change in fermionic charge/unit length associated with the wall at $x=0$ is

$$\sigma=\frac{1}{L_y}\int_0^{L_y}\mathrm{d}y\int_0^{L_x}\mathrm{d}x\,f(x)\rho(\boldsymbol{r}). \tag{3.10}$$

In (3.10) we have introduced the measurement profile function $f(x)$ (Kivelson and Schrieffer 1982, Rajaraman and Bell 1982). This is a function which is unity for $x=0$ and goes smoothly to zero as $x\to\infty$. It represents the range of measurement made on the system. In general, having the linear charge operator σ tempered by this function rather than a sharp cutoff avoids unphysical fluctuation effects. We take $f(x)$ to have the form

$$f(x)=\mathrm{e}^{-x/d}. \tag{3.11}$$

The length d is microscopically large but macroscopically small as is appropriate to a physical measurement.

The two quantities we shall calculate are

$$\langle\sigma\rangle=\langle gd|\sigma|gd\rangle \tag{3.12}$$

$$\begin{aligned}\langle(\Delta\sigma)^2\rangle&=\langle\sigma^2\rangle-\langle\sigma\rangle^2\\&=\langle gd|\sigma^2|gd\rangle-\langle gd|\sigma|gd\rangle^2.\end{aligned} \tag{3.13}$$

4. Calculation

The linear charge density operator is

$$\sigma=\frac{1}{L_y}\int_0^{L_y}\mathrm{d}y\int_0^{L_x}\mathrm{d}x\,f(x)(\psi^+(\boldsymbol{r})\psi(\boldsymbol{r})-N). \tag{4.1}$$

† The Fermi energies of the wall-free and walled systems are slightly different but this does not affect any of the results in this work.

From equation (3.2) we have

$$\psi^+(\boldsymbol{r})\psi(\boldsymbol{r})=\sum_{\substack{n_1n_2\\p_1p_2}}\frac{\exp[-\mathrm{i}y(p_1-p_2)]}{L_y}\phi_{n_1}(x)\phi_{n_2}(x)a^+_{n_1p_1}a_{n_2p_2}. \tag{4.2}$$

For the ground state (equation (3.5)) we have

$$\langle a^+_{n_1p_1}a_{n_2p_2}\rangle=\theta(\varepsilon_\mathrm{F}-\varepsilon_{n_1p_1})\delta_{n_1n_2}\delta_{p_1p_2}. \tag{4.3}$$

Hence we can write for the average linear charge density

$$\langle\sigma\rangle=\frac{1}{L_y}\int_0^{L_x}\mathrm{d}x f(x)\left(\sum_{np}|\phi_n(x)|^2\theta(\varepsilon_\mathrm{F}-\varepsilon_{np})-L_yN\right). \tag{4.4}$$

We evaluate the sum in the above equation by going to the large area limit in which it can be replaced by an integral†. Using equation (3.3) we have

$$\sum_{np}|\phi_n(x)|^2\theta(\varepsilon_\mathrm{F}-\varepsilon_{np})$$

$$=\frac{L_x}{\pi}\frac{L_y}{2\pi}\frac{2}{L_x}\int_{-\infty}^{\infty}\mathrm{d}p\int_0^{\infty}\mathrm{d}k\sin^2(kx)\theta\left(\varepsilon_\mathrm{F}-\frac{p^2+k^2}{2m}\right) \tag{4.5}$$

$$=\frac{2L_y}{\pi^2}\int_0^{p_\mathrm{F}}\mathrm{d}k(p_\mathrm{F}^2-k^2)^{1/2}\sin^2 kx. \tag{4.6}$$

In equation (4.6) we have introduced

$$p_\mathrm{F}=(2m\varepsilon_\mathrm{F})^{1/2}=(4\pi N)^{1/2} \tag{4.7}$$

the last equality following from (3.8).

Writing $\sin^2 kx=\frac{1}{2}(1-\cos 2kx)$ and using (4.7) we find

$$\sum_{np}|\phi_n(x)|^2\theta(\varepsilon_\mathrm{F}-\varepsilon_{np})=L_yN-\frac{L_y}{\pi^2}\int_0^{p_\mathrm{F}}\mathrm{d}k(p_\mathrm{F}^2-k^2)^{1/2}\cos 2kx. \tag{4.8}$$

Hence we can write (4.4) as

$$\langle\sigma\rangle=-\frac{1}{\pi^2}\int_0^{p_\mathrm{F}}\mathrm{d}k(p_\mathrm{F}^2-k^2)^{1/2}\int_0^{L_x}\mathrm{d}x f(x)\cos 2kx. \tag{4.9}$$

Replacing the upper limit of the x integral by ∞ and using the form (3.11) for $f(x)$ we find for $p_\mathrm{F}d\gg 1$

$$\langle\sigma\rangle=-\frac{p_\mathrm{F}}{4\pi}+\mathrm{O}\left(\frac{1}{p_\mathrm{F}d}\right). \tag{4.10}$$

Hence we see that in the vicinity of the wall we have an average excess charge/unit length of $-p_\mathrm{F}/4\pi$ plus vanishingly small corrections. The questions we address next is how large are the mean square deviations from this value.

The mean square deviations in the linear charge are given by

$$\langle(\Delta\sigma)^2\rangle=\langle\sigma^2\rangle-\langle\sigma\rangle^2. \tag{4.11}$$

† In approximating the sum by an integral we lose an x-independent term of relative order $(p_\mathrm{F}L_x)^{-1}$. This term corresponds to an increase in the average charge spread over the area which compensates for that squeezed out at the hard walls. By virtue of the finite measurement range d, the contribution of the omitted term to $\langle\sigma\rangle$ is down on the leading term by a factor of $\mathrm{O}(d/L_x)$ and hence is neglectable.

Squaring equation (4.1) and taking its expectation value we obtain

$$\langle\sigma^2\rangle = L_y^{-2}\int d^2r_1\, d^2r_2\, f(x_1)f(x_2)$$
$$\times(\langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_1)\psi^+(\boldsymbol{r}_2)\psi(\boldsymbol{r}_2) - 2N\langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_1)\rangle + N^2). \tag{4.12}$$

Noting that

$$\langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_1)\psi^+(\boldsymbol{r}_2)\psi(\boldsymbol{r}_2)\rangle$$
$$= \langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_1)\rangle\langle\psi^+(\boldsymbol{r}_2)\psi(\boldsymbol{r}_2)\rangle + \langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\rangle\langle\psi(\boldsymbol{r}_1)\psi^+(\boldsymbol{r}_2)\rangle \tag{4.13}$$

we can combine equations (4.11) and (4.12) to obtain

$$\langle(\Delta\sigma)^2\rangle = L_y^{-2}\int d^2r_1\, d^2r_2\, f(x_1)f(x_2)\langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\rangle\langle\psi(\boldsymbol{r}_1)\psi^+(\boldsymbol{r}_2)\rangle. \tag{4.14}$$

(The integrals are over the domain (2.1).) Using (3.7) and (4.3) we have

$$\langle\psi^+(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\rangle = \sum_{np}\phi^*_{np}(\boldsymbol{r}_1)\phi_{np}(\boldsymbol{r}_2)\theta(\varepsilon_F - \varepsilon_{np}) \tag{4.15}$$

$$\langle\psi(\boldsymbol{r}_1)\psi^+(\boldsymbol{r}_2)\rangle = \sum_{np}\phi_{np}(\boldsymbol{r}_1)\phi^*_{np}(\boldsymbol{r}_2)\theta(\varepsilon_{np} - \varepsilon_F). \tag{4.16}$$

Thus (4.14) becomes, on using orthogonality over y momenta,

$$\langle(\Delta\sigma)^2\rangle = L_y^{-2}\int_0^{L_x} dx_1 \int_0^{L_x} dx_2\, f(x_1)f(x_2)$$
$$\times\left(\sum_{n_1n_2p}\phi_{n_1}(x_1)\phi_{n_1}(x_2)\phi_{n_2}(x_1)\phi_{n_2}(x_2)\theta(\varepsilon_F - \varepsilon_{n_1p})\theta(\varepsilon_{n_2p} - \varepsilon_F)\right). \tag{4.17}$$

Replacing the sum in equation (4.17) by an integral gives

$$\sum_{n_1n_2p} = \left(\frac{L_x}{\pi}\right)^2\left(\frac{L_y}{2\pi}\right)\left(\frac{2}{L_x}\right)^2\int_0^\infty dk_1\int_0^\infty dk_2\int_{-\infty}^\infty dp\,\sin(k_1x_1)\sin(k_1x_2)\sin(k_2x_1)$$
$$\times\sin(k_2x_2)\theta(p_F^2 - k_1^2 - p^2)\theta(k_2^2 + p^2 - p_F^2) \tag{4.18}$$
$$= \left(\frac{2}{\pi}\right)^2\frac{L_y}{\pi}\int_0^{p_F} dp\int_0^{(p_F^2-p^2)^{1/2}} dk_1\int_{(p_F^2-p^2)^{1/2}}^\infty dk_2\,\sin(k_1x_1)\sin(k_1x_2)$$
$$\times\sin(k_2x_1)\sin(k_2x_2). \tag{4.19}$$

If we insert this in equation (4.17) and carry out the x integrations (with $L_x\to\infty$) we obtain

$$\langle(\Delta\sigma)^2\rangle = \left(\frac{d}{\pi}\right)^2\frac{1}{\pi L_y}\int_0^{p_F} dp\int_0^{(p_F^2-p^2)^{1/2}} dk_1\int_{(p_F^2-p^2)^{1/2}}^\infty dk_2$$
$$\times\left(\frac{1}{1+[(k_1-k_2)d]^2} - \frac{1}{1+[(k_1+k_2)d]^2}\right)^2. \tag{4.20}$$

The first term in the large brackets dominates the integral. Hence on going to dimensionless variables $z = p/p_F$, $x = k_1/p_F$, $y = k_2/p_F$ we find

$$\langle(\Delta\sigma)^2\rangle \simeq \frac{1}{p_FL_yd^2}\frac{1}{\pi^3}\int_0^1 dz\int_{(1-z^2)^{1/2}}^\infty dy\int_0^{(1-z^2)^{1/2}} dx\frac{1}{[\varepsilon^2 + (x-y)^2]^2} \tag{4.21}$$

where

$$\varepsilon = (p_F d)^{-1} \ll 1. \tag{4.22}$$

Changing to the variables $w = y - x$, $v = y - (1-z^2)^{1/2}$ the integral in equation (4.21) becomes, on integrating by parts with respect to v

$$\int_0^1 dz \int_0^\infty dv \left(\frac{v}{(\varepsilon^2 + v^2)^2} - \frac{v}{\{\varepsilon^2 + [v + (1-z^2)^{1/2}]\}^2} \right) = \frac{1}{2\varepsilon^2} + O(\ln \varepsilon). \tag{4.23}$$

Hence we can write equation (4.21) as

$$\langle (\Delta\sigma)^2 \rangle \simeq \frac{1}{2\pi^3} \frac{p_F}{L_y} \tag{4.24}$$

or combining this result with equation (4.10) for the average linear charge

$$\frac{\langle (\Delta\sigma)^2 \rangle}{\langle \sigma \rangle^2} \simeq \frac{8}{\pi} \frac{1}{p_F L_y}. \tag{4.25}$$

We can interpret this result in two equivalent ways. It is the (fractional deviation)^2 of either

(i) the (excess) linear charge density of a system with boundary length L_y, or

(ii) the (excess) boundary charge of a system with boundary length L_y.

Thus since p_F^{-1} is a microscopic length we will have vanishingly small deviations from the mean value of (say) the linear charge density provided L_y is a macroscopic length. The whole stabilisation of the linear charge density results from the huge disparity of the two fundamental length scales in the problem.

The result of equation (4.25) suggests that in one dimension the fractional deviations of the charge (not charge density) at the boundary are not small. We cannot directly infer what values these have from equation (4.25) by taking $L_y \to 0$ since we assumed it large at various intermediate stages of the calculation. It is straightforward though to repeat the above calculation in one dimension and we find for the charge excess

$$\langle Q \rangle = -\frac{1}{4} + O\left(\frac{1}{p_F d}\right) \tag{4.26}$$

and the deviations

$$\langle (\Delta Q)^2 \rangle = \frac{1}{2\pi^2} \sim \langle Q \rangle^2. \tag{4.27}$$

It is, perhaps, interesting to return to the two-dimensional system and consider what charge at the boundary is associated with particles having definite transverse (i.e. parallel to the wall) momentum p_y. By similar methods to those already presented we find

$$\langle Q(p_y) \rangle = -\frac{1}{2\pi} \theta(p_F^2 - p_y^2) \tan^{-1}[2d(p_F^2 - p_y^2)^{1/2}] \tag{4.28}$$

$$\langle (\Delta Q)^2(p_y) \rangle \simeq \frac{1}{2\pi^2} \theta(p_F^2 - p_y^2) \left\{ \frac{a^2}{\varepsilon^2 + a^2} + O\left[\left(\frac{\varepsilon}{a}\right)^2\right] \right\} \tag{4.29}$$

with

$$a = (1 - p_y^2/p_F^2)^{1/2} \qquad \varepsilon = (p_F d)^{-1}.$$

Hence we see that except for momenta within $\sim d^{-1}$ of the Fermi momentum we have the one-dimensional results of equations (4.26) and (4.27). Thus the two-dimensional system may be considered to be a sum of many one-dimensional systems.

5. Discussion

In this work we have considered some of the properties of very simple non-relativistic fermion systems. We have shown that in two-dimensional systems hard wall boundary conditions have the results of producing a linear charge density at the boundary which has the property of being essentially fluctuation free†.

It was, as the reader may well have guessed, the one-dimensional system which inspired this work. It is an intriguing possibility that a related system could exhibit (essentially fluctuation free) fractional charge effects. Such a system would require interactions between the fermions leading to a ground state with correlations beyond those produced by Fermi statistics. The mean square charge deviations in this case can be written as

$$\langle(\Delta Q)^2\rangle = \int_0^L \mathrm{d}x_1 \int_0^L \mathrm{d}x_2 f(x_1) f(x_2) C(x_1, x_2) \tag{5.1}$$

with

$$C(x_1, x_2) = \langle\psi^+(x_1)\psi(x_1)\psi^+(x_2)\psi(x_2)\rangle - \langle\psi^+(x_1)\psi(x_1)\rangle\langle\psi^+(x_2)\psi(x_2)\rangle. \tag{5.2}$$

It would seem to put some stringent conditions on $C(x_1, x_2)$ and $\langle\psi^+(x_1)\psi(x_1)\rangle$ if both a sizable charge and small fractional deviations can be produced.

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† For fermions in dimension $d(\geqslant 2)$ the charge/$(d-1)$-dimensional area will also have this property.